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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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UNSTABLE CAPILLARY WAVES ON SURFACE OF SEPARATION OF TWO VISCOUS FLUIDS

By V. A. Borodin and Y. F. Dityakin

Translation

"Neustoichivye Kapilliarnye Volny na Poverkhnosti Razdela Dvukh Vyazkikh Zhidkostei." Prikladnaya Matematika i Mekhanika. Vol. XIII, no. 3, 1949.

Washington
April 1951

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UNSTABLE CAPILLARY WAVES ON SURFACE OF SEPARATION OF TWO VISCOUS FLUIDS *

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The study of the breakup of a liquid jet moving in another medium, for example, a jet of fuel from a nozzle, shows that for sufficiently large outflow velocities the jet breaks up into a certain number of drops of different diameters. At still larger outflow velocities, the continuous part of the jet practically vanishes and the jet immediately breaks up at the nozzle into a large number of droplets of varying diameters (the case of "atomization"). The breakup mechanism in this case has a very complicated character and is quite irregular, with the droplets near the nozzle forming a divergent cone.

Rayleigh (reference 1) was the first to make a theoretical study of the jet and to establish the possibility of droplet formation. The disturbance of a jet of an ideal fluid flowing into a vacuum and having a wave length 4.4 times as large as the diameter of the jet is shown to grow more rapidly than other disturbances; eventually, the jet breaks up into droplets of the same diameter. Rayleigh succeeded in determining theoretically the drop diameter, the value of which agrees well with tests on jets issuing with very small velocities. Later, the viscosity of the jet was also taken into consideration. The viscosity is found to decrease the rate of amplitude increase of the disturbances but the ratio of the optimal length of the wave to the diameter of the jet remains unchanged.

Other authors that studied the conditions of the axial-symmetrical breakup of a jet of a viscous liquid found that the ratio of the optimal wave length to the jet diameter was somewhat greater than that computed by Rayleigh.

In addition to the viscosity, Tomotika (reference 2) took into account the density and viscosity of the medium surrounding the jet and obtained good agreement with tests on jets issuing with very small velocities for which droplets of the same diameter are formed.

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*"Neustoichivye Kapilliarnye Volny na Poverkhnosti Razdela Dvukh Vyazkikh Zhidkostei." Prikladnaya Matematika i Mekhanika. Vol. XIII, no. 3, 1949, pp. 267-276.

Neither of the aforementioned theories of the breakup of a liquid jet provided a basis for the phenomenon for the case of breakup into droplets of different diameter, a fact that is explained by the idealized conditions of the problem. This idealization consisted either in neglecting the viscosity of the jet, the density, and viscosity of the surrounding medium, or the inertial forces. Such simplifications were assumed in view of the complicated mathematical equation (generally transcendental) that determines the relation between the wavelength and the increment of the vibration amplitude.

In the present paper, an attempt is made to provide a mathematical basis for the possibility of the appearance of droplets of different diameters as a result of the jet breakup on the basis of the consideration of unstable capillary waves on the surface of separation of two viscous liquids.

For simplification of the solution of the problem, particularly for obtaining the algebraic characteristic of the equation, the lengths of the capillary waves on the surface of the liquid jet are assumed to be so small in comparison with the jet radius that the jet may be considered infinitely large; study of the stability of the plane surface of separation of two infinitely extending viscous fluids can thus be made. This assumption represents a considerable degree of idealization but nevertheless permits a qualitative explanation of not one but several unstable capillary waves that, in passing through the jet, lead to the formation of droplets of differing diameters.

The existence of several unstable capillary waves is demonstrated that can lead to the breakaway of several infinitely long strings of different dimensions from the partition surface. The problem investigated gives a rough approximation of the disintegration pattern of a liquid jet in another medium and does not pretend to explain the complicated mechanism of the limiting form of the disintegration of a jet, namely, atomization. Nevertheless, one of the peculiarities of atomization, the appearance of a dimension spectrum of the droplets, begins to appear even for the given idealized consideration of the stability of the partition surface.

1. Equations of small waves and their solution. - A plane surface of separation of two infinitely extending viscous fluids (fig. 1) is considered. The viscosity and density of the lower fluid are denoted by μ_1 and ρ_1, respectively, and of the upper fluid by μ_2 and ρ_2. The lower fluid is assumed to move with the velocity V_1 and the upper fluid with the velocity V_2, the direction of motion being the same and the velocities independent of y.

A study of the character of the equilibrium of the surface of separation under the action of the viscous forces and the forces of surface tension that impart to both liquids small disturbances parallel to the x-axis is presented. The fluids shall be considered incompressible and weightless and shall cause certain disturbances to the components of the motion.

$$V_x = V + v_x$$

$$V_y = v_y$$

$$p = P + p^*$$

It is further assumed that the velocities of the imposed disturbances and their derivatives up to the third inclusive are small and that the magnitudes of the second-order smallness may be neglected.

From the Navier-Stokes equations, the following equations of the imposed disturbances are obtained:

$$\left.\begin{aligned} \frac{\partial v_x}{\partial t} + V\frac{\partial v_x}{\partial x} &= -\frac{1}{\rho}\frac{\partial p^*}{\partial x} + \nu\Delta v_x \\ \frac{\partial v_y}{\partial t} + V\frac{\partial v_y}{\partial x} &= -\frac{1}{\rho}\frac{\partial p^*}{\partial y} + \nu\Delta v_y \end{aligned}\right\} \tag{1.1}$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

The equation of continuity is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{1.2}$$

By introducing the stream function of the disturbance

$$v_x = \frac{\partial\psi}{\partial y} \qquad v_y = -\frac{\partial\psi}{\partial x} \tag{1.3}$$

and by eliminating the pressure p^* from equations (1.1), the idealized equation is thus obtained in the Helmholtz form

$$\nu\Delta\Delta\psi - V\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\Delta\psi}{\partial t} = 0 \tag{1.4}$$

Let the stream function of the imposed disturbance be a periodic function of x and of the time t:

$$\psi = f(y)e^{i(\alpha x - \beta t)} \qquad \left(\alpha = \frac{2\pi}{\lambda}\right) \tag{1.5}$$

where α is the propagated circular frequency of the vibrations (the wave number), λ is the wavelength of the imposed disturbance, $\beta = \beta_r + i\beta_i$ is the complex frequency of vibrations in time, β_r is the real frequency of vibration in time, and β_i is the increment of the growth of vibration or the decrement of damping.

The character of the wave motion on the surface of separation after the imparting of disturbances to both surfaces will thus depend on the sign of the imaginary part of the frequency β_i. If β_i is positive, there will be an increase in the wave amplitude with time; if β_i is negative, there will be a damping of the wave amplitude; finally, if $\beta_r = 0$, there will be an aperiodic increase $(\beta_i > 0)$ or a damping $(\beta_i < 0)$ of the wave amplitude. By substituting expression (1.5) in equation (1.4), the following equation is obtained:

$$\nu f^{IV} - (2\alpha^2\nu - i\beta) f'' - (i\beta\alpha^2 - \nu\alpha^4) f - iV\alpha (f'' - \alpha^2 f) = 0 \tag{1.6}$$

The problem of the characteristic values of a homogeneous system of equations of the fourth order will be considered.

By setting $f'' - \alpha^2 f = \varphi$, a system of equations of the second order is obtained.

$$\varphi'' + \left(i\,\frac{\beta - V\alpha}{\nu} - \alpha^2\right)\varphi = 0 \qquad f'' - \alpha^2 f = \varphi \tag{1.7}$$

Hereinafter, the following notations are introduced:

$$\sqrt{i\,\frac{\beta - V_1\alpha}{\nu_1} - \alpha^2} = m_1 \qquad \sqrt{i\,\frac{\beta - V_2\alpha}{\nu_2} - \alpha^2} = m_2 \tag{1.8}$$

The solution of the first of equations (1.7) has the form

$$\varphi = C_1 e^{im_1 y} + C_2 e^{-im_1 y} \tag{1.9}$$

By substituting expression (1.9) in the second of equations (1.7), a non-homogeneous equation is obtained for which the solution is

$$f = -\frac{e^{im_1y}}{m_1^2 + \alpha^2} C_1 - \frac{e^{-im_1y}}{m_1^2 + \alpha^2} C_2 + e^{\alpha y} C_3 + e^{-\alpha y} C_4 \tag{1.10}$$

The stream function for the lower and upper liquids according to equation (1.5) will be

$$\psi_1 = e^{i(\alpha x-\beta t)} \left(-\frac{e^{im_1y}}{m_1^2 + \alpha^2} C_1 - \frac{e^{-im_1y}}{m_1^2 + \alpha^2} C_2 + e^{\alpha y} C_3 + e^{-\alpha y} C_4\right) \tag{1.11}$$

$$\psi_2 = e^{i(\alpha x-\beta t)} \left(-\frac{e^{im_2y}}{m_2^2 + \alpha^2} C_5 - \frac{e^{-im_2y}}{m_2^2 + \alpha^2} C_6 + e^{\alpha y} C_7 + e^{-\alpha y} C_8\right) \tag{1.12}$$

The arbitrary constants C_i must be determined from the conditions on the surface of separation and at infinity.

2. Boundary conditions. - The boundary conditions of the problem will be as follows:

1. At infinity ($y = \pm\infty$), finite solutions must be maintained for ψ_1 and ψ_2. Hence, the arbitrary constants of the terms with positive exponents for ψ_1 and with negative exponents for ψ_2 must be equated to zero: $C_1 = C_3 = C_6 = C_8 = 0$. Thus, equations (1.10) and (1.11) will have the form

$$\left.\begin{aligned} \psi_1 &= e^{i(\alpha x-\beta t)} \left(-\frac{e^{-im_1y}}{m_1^2 + \alpha^2} C_2 + e^{-\alpha y} C_4\right) \\ \psi_2 &= e^{i(\alpha x-\beta t)} \left(-\frac{e^{im_2y}}{m_2^2 + \alpha^2} C_5 + e^{\alpha y} C_7\right) \end{aligned}\right\} \tag{2.1}$$

2. On the surface of separation, there must be no slip, that is,

$$\left.\begin{aligned} \left(v_{x1}\right)_{y=0} &= \left(v_{x2}\right)_{y=0} \\ \left(v_{y1}\right)_{y=0} &= \left(v_{y2}\right)_{y=0} \end{aligned}\right\}$$

or

$$\left.\begin{aligned}\left(\frac{\partial\psi_1}{\partial y}\right)_{y=0} &= \left(\frac{\partial\psi_2}{\partial y}\right)_{y=0}\\ \left(\frac{\partial\psi_1}{\partial x}\right)_{y=0} &= \left(\frac{\partial\psi_2}{\partial x}\right)_{y=0}\end{aligned}\right\} \tag{2.2}$$

3. The tangential stresses on the surface of separation are continuous

$$\mu_1(\Delta\psi_1)_{y=0} = \mu_2(\Delta\psi_2)_{y=0} \tag{2.3}$$

4. The difference between the normal stresses p_{y1} and p_{y2} on the surface of separation is equal to the pressure brought about by the surface tension; that is,

$$p_{y1} - p_{y2} = \left(- p_1 + 2\mu_1 \frac{\partial v_{y1}}{\partial y}\right) - \left(- p_2 + 2\mu_2 \frac{\partial v_{y2}}{\partial y}\right) = - T \frac{\partial^2 h}{\partial x^2} \tag{2.4}$$

where T is the capillary constant of one liquid relative to the other and h is the rise in the surface of separation at the point x.

By using equations (2.1), the boundary conditions (2.2) are obtained in the form

$$\left.\begin{aligned}\frac{im_1}{m_1^2 + \alpha^2} C_2 - \alpha C_4 + \frac{im_2}{m_2^2 + \alpha^2} C_5 - \alpha C_7 = 0\\ -\frac{C_2}{m_1^2 + \alpha^2} + C_4 + \frac{C_5}{m_2^2 + \alpha^2} - C_7 = 0\end{aligned}\right\} \tag{2.5}$$

Similarly, the boundary condition (2.3) is obtained in the form

$$\frac{\mu_1}{\mu_2}\left[-\left(-\frac{C_2}{m_1^2 + \alpha^2} + C_4\right)\alpha^2 + \frac{m_1^2}{m_1^2 + \alpha^2} C_2 + \alpha^2 C_4\right]$$

$$= -\left(-\frac{C_5}{m_2^2 + \alpha^2} + C_7\right)\alpha^2 + \frac{m_2^2}{m_2^2 + \alpha^2} C_5 + \alpha^2 C_7 \tag{2.6}$$

The pressures p_1 and p_2 are computed from equations (1.1) and (2.1). Thus

$$p_1 = \rho_1 e^{i(\alpha x-\beta t)}\left[\frac{\alpha e^{-im_1 y}}{m_1{}^2 + \alpha^2}\left(\frac{\nu_1\alpha^2}{m_1} - \frac{iV_1\alpha}{m_1} + \frac{i\beta}{m_1} + m_1\nu_1\right) C_2 + \left(\beta + i\nu_1\alpha^2 - V_1\alpha - i\nu_1\alpha^2\right)e^{-\alpha y} C_4\right] \tag{2.7}$$

$$p_2 = \rho_2 e^{i(\alpha x-\beta t)}\left[\frac{\alpha e^{im_2 y}}{m_2{}^2 + \alpha^2}\left(\frac{\nu_2\alpha^2}{m_2} - \frac{iV_2\alpha}{m_2} + \frac{i\beta}{m_2} + m_2\nu_2\right)C_5 + (\beta - V_2\alpha)\ e^{\alpha y}\ C_7\right]$$

The rise of a point on the surface is a periodic function of x and t.

$$h = He^{i(\alpha x-\beta t)} \tag{2.8}$$

where H is the maximal rise of a point on the surface of separation.

The velocity of the raised point on the surface of separation is

$$-\left(v_{y1}\right)_{y=0} = \left(\frac{\partial\psi_1}{\partial x}\right)_{y=0} = \frac{\partial h}{\partial t} + V_1\,\frac{\partial h}{\partial x} \tag{2.9}$$

After differentiating expressions (2.1) and (2.8) and by substituting in expression (2.9), the following equation is obtained:

$$H = \frac{\alpha}{\alpha V_1 - \beta}\left(C_4 - \frac{C_2}{m_1{}^2 + \alpha^2}\right) \tag{2.10}$$

By substituting equation (2.10) in (2.9) and by differentiating equation (2.9),

$$\frac{\partial^2 h}{\partial x^2} = \frac{\alpha^3}{\alpha V_1 - \beta}\left(\frac{C_2}{m_1{}^2 + \alpha^2} - C_4\right)e^{i(\alpha x-\beta t)} \tag{2.11}$$

By computing the derivatives $\partial v_{y1}/\partial y$ and $\partial v_{y2}/\partial y$ and substituting them simultaneously with expressions (2.7) and (2.11) in (2.4), the following boundary condition is obtained:

$$\frac{\alpha}{m_1{}^2 + \alpha^2}\left[\frac{-\mu_1\alpha^2 + i\rho_1 V_1\alpha - i\beta\rho_1}{m_1} + m_1\mu_1 + \frac{T\alpha^2}{\alpha V_1 - \beta}\right]C_2 -$$

$$\left(\rho_1\beta - \rho_1 V_1\alpha - i2\mu_1\alpha^2 + \frac{T}{\alpha V_1 - \beta}\right)C_4 + \frac{\alpha}{m_2{}^2 + \alpha^2}\left[\frac{\mu_2\alpha^2 - i\rho_2 V_2\alpha + i\beta\rho_2}{m_2} + \right.$$

$$\left. 3m_2\mu_2\right]C_5 + \left(\rho_2\beta - \rho_2 V_2\alpha + i2\mu_2\alpha^2\right)C_7 = 0 \tag{2.12}$$

The following nondimensional parameters are then introduced:

$$\left.\begin{aligned} Z &= \frac{c}{\nu_1\alpha} \\ R_1 &= \frac{V_1}{\nu_1\alpha} \\ R_2 &= \frac{V_2}{\nu_2\alpha} \\ A &= \frac{\nu_1}{\nu_2} \\ N &= \frac{T\rho_1}{\mu_1{}^2\alpha} \\ K &= \frac{\mu_1}{\mu_2} \end{aligned}\right\} \tag{2.13}$$

where $c = \beta/\alpha$ is the complex wave velocity.

Equations (1.8) can then be represented in the forms

$$m_1 = \alpha\sqrt{i(Z - R_1) - 1}$$

$$m_2 = \alpha\sqrt{i(ZA - R_2) - 1}$$

Equations (2.5), (2.6), and (2.12) are represented in nondimensional parameters. The following notations are first introduced:

$$\left.\begin{aligned}
a_1{}^* &= \mu_1 \frac{i}{Z-R_1}\left[\frac{2}{\sqrt{i(Z-R_1) - 1}} + \frac{N}{Z-R_1}\right] = \mu_1 a_1 \\
b_1{}^* &= \alpha^2\mu_1\left(Z-R_1-2i + \frac{N}{R_1-Z}\right) = \alpha^2\mu_1 b_1 \\
c_1{}^* &= \mu_2 \frac{2(i - 2R_2 + 2AZ)}{(ZA-R_2)\sqrt{i(ZA-R_2) - 1}} = \mu_2\alpha^2 c_1 \\
a_3{}^* &= -\frac{1}{\alpha^2}\frac{i}{Z-R_1} = \frac{a_3}{\alpha^2} \\
d_1{}^* &= \mu_2\alpha^2\ (ZA-R_2 + 2i) = \mu_2\alpha^2 d_1 \\
c_3{}^* &= -\frac{1}{\alpha^2}\frac{i}{ZA-R_2} = \frac{c_3}{\alpha^2} \\
a_2{}^* &= \frac{1}{\alpha}\sqrt{i(Z-R_1) - 1} = \frac{a_2}{\alpha} \\
c_2{}^* &= \frac{1}{\alpha}\frac{\sqrt{i(ZA-R_2) - 1}}{ZA-R_2} = \frac{c_2}{\alpha}
\end{aligned}\right\} \qquad (2.14)$$

where a_1, a_2, a_3, b_1, c_1, c_2, c_3, and d_1 are likewise nondimensional magnitudes.

The following system of equations is then obtained for the constants C_2, C_4, C_5, and C_7:

$$\left.\begin{aligned}
a_1{*}C_2 - b_1{*}C_4 + c_1{*}C_5 + d_1{*}C_7 &= 0 \\
a_2{*}C_2 - \alpha C_4 + c_2{*}C_5 - \alpha\ C_7 &= 0 \\
-\alpha_3{*}C_2 + C_4 + c_3{*}C_5 - C_7 &= 0 \\
K\ C_2 - C_5 &= 0
\end{aligned}\right\} \tag{2.15}$$

This system of homogeneous equations has solutions different from zero if its determinant is equal to zero. By setting up the determinant and expanding

$$2K(a_1 + c_1) + (d_1 - Kb_1)(a_2 + Kc_2) + (Kb_1 + d_1)(Kc_3 - a_3) = 0$$

By solving this equation for Z, the following wave equation of the 18th degree with complex coefficients is obtained:

$$r_{18}Z^{18} + (r_{17} + is_{17})\ Z^{17} + \ldots + (r_1 + is_1)\ Z + (r_0 + is_0) = 0 \tag{2.16}$$

The real and imaginary parts of the coefficients depend on the five nondimensional parameters: R_1, R_2, A, N, and K.

3. Investigation of roots of characteristic equation. - The increase in oscillation, that is, the loss of stability of the surface of separation, arises from those waves for which the imaginary part of the frequency is positive ($\beta_i > 0$). Hence, the investigation of the roots of equation (2.16) should determine those ranges of the parameter N or the wave number α in which the complex roots of the equation lie in the upper half-plane.

By the Rayleigh hypothesis, the further development of an unstable deformation, that is, the form and dimensions of the parts breaking away, is determined by the critical (or optimal) disturbances. The critical disturbances may be defined as those that develop more rapidly than the others or that correspond to the maximum increment of the growth β_i. This principle of determining the character of the unstable deformations by the character of the maximum unstable disturbance has been experimentally confirmed by a number of investigators (reference 3).

In the case considered, the growth in the amplitudes of the oscillations will lead to breakaway of infinitely long strings from the surface of separation, similar to the formation and breakaway of wave crests. The separation will take place for such values of α or wavelengths λ for which β_1 has the maximal value.

If a spectrum of small-period disturbances that can be developed into a Fourier series can be assumed to be imposed on both liquids, the harmonics with the wavelengths equal to the wavelengths of the maximal unstable disturbances bring about a separation of infinitely long strings from the partition surface. Because the characteristic dimension (for example, the diameter of the transverse string) is connected with the length of maximal unstable disturbance, strings of different dimensions will break away from the surface of separation. In figure 2, the scheme of formation of such strings for three successive instants of time is shown.

Investigation of the roots of the simplest particular case of equation (2.16) is presented.

Let both fluids be stationary and their kinetic viscosities the same. In this case, $V_1 = V_2 = 0$, $\nu_1 = \nu_2$, $m_1 = m_2$, $A = 1$, $R_1 = R_2 = 0$, and equation (2.16) goes over into an equation of the 8th degree whose coefficients depend only on the two parameters K and N:

$$A_0Z^8 + (A_1 + iB_1)Z^7 + (A_2 + iB_2)Z^6 + (A_3 + iB_3)Z^5 + (A_4 + iB_4)Z^4 + (A_5 + iB_5)Z^3 + (A_6 + iB_6)Z^2 + (A_7 + iB_7)Z + A_8 = 0 \qquad (3.1)$$

where

$$A_0 = -(1 - K)^2$$

$$A_2 = 2K(K - 1)N - K^4 + 2K^3 - 4K^2 + 6K + 13$$

$$A_1 = -2K(K - 1)^2$$

$$A_3 = 4K^2(K - 1)N - 2K(3K^2 + 13)$$

$$A_7 = 2K^3N^2$$

$$A_4 = -K^2N^2 + 2(K^4 - K^3 + 3K^2 + 5K)N - 12K^3 + 26K^2 - 10K - 9$$

$$A_5 = -2K^3N^2 + 12K^3N - 8K^3 - 8K^2$$

$$A_8 = -K^2(1 + 2K)N^2$$

$$A_6 = (1 - K^2)K^2N^2 + (4K^4 - 10K^3 - 4K^2 - 6K)N - 8K^3 + 12K^2 - 8K + 4$$

$$B_1 = 2(K^2 + 2K - 3)$$

$$B_3 = 3K^4 - 14K^3 + 13K^2 + 18K + 13 - 8KN$$

$$B_2 = 4K(K - 1)^2$$

$$B_4 = 8K^3 - 4K^2 - 20K - 2K^3N$$

$$B_6 = 4K^2(1 - K)N$$

$$B_5 = -2K^2N^2 + \left[2K(1 + K)(1 + K - K^2) + 8K^3 + 4K^2 + 4K\right]N + 4(K - 1 - K^2)(1 + K - K^2) + 4K^4 - 20K^3 - 4(K - 1 - K^2)^2$$

$$B_7 = \left[K^2(1 + K)^2 - 2K^4\right]N^2 + \left[4K^4 + 4K(1 + K)(K - 1 - K^2)\right]N \tag{3.2}$$

The characteristic equation (3.1) is a polynomial whose coefficients depend nonlinearly on the two parameters K and N. Each pair of values of the parameters K and N or each point of the plane KN correspond to the completely defined polynomial (3.1), that is, completely determined values of the eight roots of the polynomial. In the plane KN, it is evidently possible to find a curve, each point of which corresponds to the polynomial (3.1), that has at least one root located on the real axis so that only in crossing this curve is a crossing of the roots through the real axis possible. This curve breaks up the plane KN into regions, the points of which each correspond to polynomials (3.1), that have the same number of roots with positive imaginary part.

These curves are constructed by making use of the method of Y. I. Neimark (reference 4) that permits a breakup of the plane of the parameters for the roots of the polynomial lying in the left or right half-plane.

The substitution $Z = -i\zeta$ is made. The upper half-plane of the roots of equation (3.1) is transformed into the left half-plane of the roots of the equation

$$- A_0\zeta^8 + i(A_1 + iB_1)\zeta^7 + (A_2 + iB_2)\zeta^6 - i(A_3 + iB_3)\zeta^5 - (A_4 + iB_4)\zeta^4 + i(A_5 + iB_5)\zeta^3 + (A_6 + iB_6)\zeta^2 - (A_7 + iB_7)\zeta + A_8 = 0 \tag{3.3}$$

By substituting $\zeta = i\xi/\eta$ in the preceding equation and multiplying the result by η^8, equation (3.3) is reduced to the form

$$F(\xi,\eta) + iG(\xi,\eta) = 0 \tag{3.4}$$

where

$$F(\xi,\eta) = A_0\xi^8 + A_1\xi^7\eta + A_2\xi^6\eta^2 + A_3\xi^5\eta^3 + A_4\xi^4\eta^4 + A_5\xi^3\eta^5 + A_6\xi^2\eta^6 + A_7\xi\eta^7 + A_8\eta^8 \tag{3.5}$$

$$G(\xi,\eta) = B_1\xi^7\eta + B_2\xi^6\eta^2 + B_3\xi^5\eta^3 + B_4\xi^4\eta^4 + B_5\xi^3\eta^5 + B_6\xi^2\eta^6 + B_7\xi\eta^7$$

If R_{2n} is the space of complex polynomials of degree n and $D(k,n-k)$ is the manifold of polynomials R_{2n} having k roots to the left and $n - k$ roots to the right of the imaginary axis of the complex sphere, then by setting up the following table:

$$\left\{\begin{matrix} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 \\ 0 & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 \end{matrix}\right\} \tag{3.6}$$

and by making the transformation

$$\left\{\begin{matrix} A_0 + \lambda_1 B_1 & A_1 + \lambda_1 B_2 & A_2 + \lambda_1 B_3 & A_3 + \lambda_1 B_4 \ldots A_8 \\ 0 & B_1 & B_2 & B_3 \quad \ldots 0 \end{matrix}\right\} \tag{3.7}$$

table (3.7) is found to correspond to a polynomial of the same type with respect to the distribution of the roots relative to the imaginary axis, as in equation (3.4).

From table (3.6), an inequality is obtained that defines the region in the plane KN corresponding to the presence of the first root of equation (3.1) in the upper half-plane:

$$A_0 B_1 < 0 \tag{3.8}$$

By setting $\lambda_1 = -A_0/B_1$ in table (3.7)

$$\left\{\begin{matrix} (A_1B_1 - A_0B_2)/B_1 & (A_2B_1 - A_0B_3)/B_1 & (A_3B_1 - A_0B_4)/B_1 & \ldots A_7 \; A_8 \\ B_1 & B_2 & B_3 & \ldots B_7 \; 0 \end{matrix}\right\} \tag{3.9}$$

Because $A_1B_1 - A_0B_2 = -16K(K-1)^3 < 0$ for $K > 1$, by multiplying the elements of the first rows of (3.9) by $B_1{}^2/(A_1B_1 - A_0B_2)$ and changing signs in the second row

$$\left\{\begin{matrix} B_1 & D_1 & D_2 & D_3 & D_4 & D_5 & D_6 & D_7 \\ -B_1 & -B_2 & -B_3 & -B_4 & -B_5 & -B_6 & -B_7 & 0 \end{matrix}\right\} \tag{3.10}$$

where

$$\left.\begin{aligned}
\frac{B_1(A_2B_1 - A_0B_3)}{A_1B_1 - A_0B_2} &= D_1 \\
\frac{B_1(A_3B_1 - A_0B_4)}{A_1B_1 - A_0B_2} &= D_2 \\
\frac{B_1(A_4B_1 - A_0B_5)}{A_1B_1 - A_0B_2} &= D_3 \\
\frac{B_1(A_5B_1 - A_0B_6)}{A_1B_1 - A_0B_2} &= D_4 \\
\frac{B_1(A_6B_1 - A_0B_7)}{A_1B_1 - A_0B_2} &= D_5 \\
\frac{B_1A_7}{A_1B_1 - A_0B_2} &= D_6 \\
\frac{B_1A_8}{A_1B_1 - A_0B_2} &= D_7
\end{aligned}\right\} \tag{3.11}$$

The first row of table (3.10) is left unchanged but to the second row is added the first row. Thus

$$\left\{\begin{matrix} B_1 & D_1 & D_2 & D_3 & D_4 & D_5 & D_6 & D_7 \\ 0 & D_1 - B_2 & D_2 - B_3 & D_3 - B_4 & D_4 - B_5 & D_5 - B_6 & D_6 - B_7 & C_7 \end{matrix}\right\} \tag{3.12}$$

From the preceding calculations, an inequality is obtained that defines the region in the plane KN that corresponds to the presence of the second root of equation (3.1) in the upper half-plane

$$B_1(D_1 - B_2) < 0 \tag{3.13}$$

By carrying out a transformation, similar to (3.7) of table (3.12),

$$\left\{\begin{matrix} B_1 + \lambda_2(D_1 - B_2) & D_1 + \lambda_2(D_2 - B_3) & D_2 + \lambda_2(D_3 - B_4)\ldots D_7 \\ 0 & D_1 - B_2 & D_2 - B_3 \quad \ldots C_7 \end{matrix}\right\} \tag{3.14}$$

By setting $\lambda_2 = - B_1/(D_1 - B_2)$ and substituting in (3.14)

$$\left\{\begin{matrix} \dfrac{D_1(D_1 - B_2) - B_1(D_2 - B_3)}{D_1 - B_2} & \dfrac{D_2(D_1 - B_2) - B_1(D_3 - B_4)}{D_1 - B_2} \\ D_1 - B_2 & D_2 - B_3 \\ \multicolumn{2}{c}{\dfrac{D_3(D_1 - B_2) - B_1(D_4 - B_5)}{D_1 - B_2} \ldots} \\ \multicolumn{2}{c}{D_3 - B_4 \ldots\ldots} \end{matrix}\right\} \tag{3.15}$$

Because $D_1(D_1 - B_2) - B_1 \ (D_2 - B_3) > 0$ for $K > 1$, by multiplying the elements of the first row of (3.15) by $(D_1 - B_2)^2/ \left[D_1(D_1 - B_2) - B_1(D_2 - B_3)\right]$

$$\left\{\begin{matrix} D_1 - B_2 & \dfrac{\left[D_2(D_1 - B_2) - B_1(D_3 - B_4)\right](D_1 - B_2)}{D_1(D_1 - B_2) - B_1(D_2 - B_3)} \ldots \\ D_1 - B_2 & D_2 - B_3 \quad \ldots \end{matrix}\right\} \tag{3.16}$$

The elements of the first row are subtracted from the elements of the second row of table (3.16).

$$\left\{\begin{matrix} D_1 - B_2 & \left[(D_2 \ (D_1 - B_2) - B_1(D_3 - B_4)\right] (D_1 - B_2) & \ldots \\ 0 & (D_2 - B_3) - \dfrac{\left[D_2(D_1 - B_2) - B_1(D_3 - B_4)\right] (D_1 - B_2)}{D_1(D_1 - B_2) - B_1(D_2 - B_3)} & \ldots \end{matrix}\right\} \tag{3.17}$$

From the preceding table, an inequality is obtained that defines the region in the plane of the parameters KN that corresponds to the presence of the third root of equation (3.1) in the upper half-plane.

$$(D_1 - B_2)\left[(D_2 - B_3) - \frac{\left[D_2(D_1 - B_2) - B_1(D_3 - B_4)\right] (D_1 - B_2)}{D_1(D_1 - B_2) - B_1(D_2 - B_3)}\right] < 0 \tag{3.18}$$

Similar conditions can be obtained for all the remaining roots of equation (3.1). This investigation has been limited to the three conditions that are sufficient for proving the existence of several unstable waves.

By replacing inequalities (3.8), (3.13), and (3.18) by equations, the equations of the curves determining the breakup of the KN plane into regions are obtained. The most interesting case of large $K = \mu_1/\mu_2 \gg 1$ is considered. From inequalities (3.8), (3.13), and (3.18) and by considering equations (3.2) and (3.11) and neglecting small powers of K, the following equations are obtained:

$$\left.\begin{aligned} 2(K-1)^3(K+3) &= 0 \\ e_0N^3 + e_1N^2 + e_2N + e_3 &= 0 \\ 4(K^2-1)(K+3)N + K(K-1)(K^3+17K^2-96K+99) &= 0 \end{aligned}\right\} \quad (3.19)$$

where

$$e_0 = 128(K^4 - K^3 - 23K^2 - 39K - 18)$$

$$e_1 = 592K(K^5 + 8.4K^4 + 3.18K^3 - 96K^2 - 20.3K + 0.98)$$

$$e_2 = 9K^2(K^6 + 8.4K^5 - 97.3K^4 - 2045K^3 + 1700K^2 + 390K + 363)$$

$$e_3 = 24K^5(K^5 + 12.3K^4 + 306K^3 - 4100K^2 + 12,300K - 7000)$$

By plotting the curves (3.19) in the KN plane and separating by hatched lines the regions corresponding to the signs of the inequalities (3.8), (3.13), and (3.18), the diagram shown in figure 3 is obtained. This diagram shows that for $K > 0$ and $N > 0$ a region of values of K and N exists that corresponds to the presence of three roots with positive imaginary part, that is, of three unstable waves on the surface of separation.

The division of the KN plane for the remaining roots could establish regions with a still greater number of roots with positive imaginary part. The given incomplete diagram already shows, however, the existence of several unstable waves. In the presence of a maximum β_i or c_i, several infinitely long strings will break away from the surface of separation, the cross-sectional dimensions of which will depend on the wavelength of the critical disturbance.

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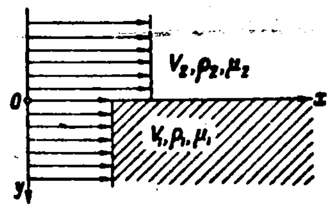

Figure 1.

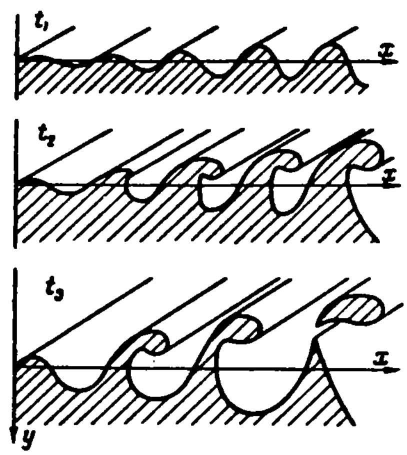

Figure 2.

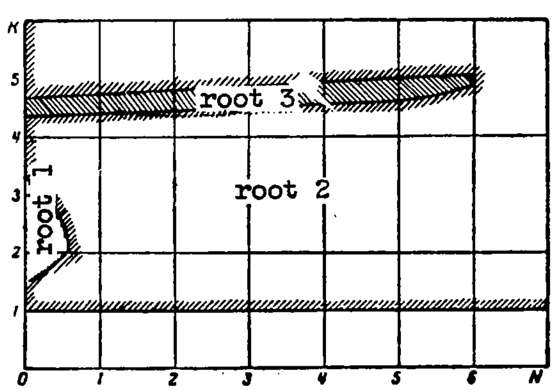

Figure 3.